

Integration of Lie bialgebroids *

KIRILL C. H. MACKENZIE

School of Mathematics and Statistics

University of Sheffield

Sheffield, S3 7RH, England

email: K.Mackenzie@sheffield.ac.uk

and

PING XU †

Department of Mathematics

The Pennsylvania State University

University Park, PA 16802, USA

email: ping@math.psu.edu

December 21, 1997

Abstract

We prove that under certain mild assumptions a Lie bialgebroid integrates to a Poisson groupoid. This includes, in particular, a new proof of the existence of local symplectic groupoids for any Poisson manifold, a theorem of Karasev and of Weinstein.

1 Introduction

A symplectic realization of a Poisson manifold P is a Poisson map from a symplectic manifold X to P which is a surjective submersion. The idea of finding symplectic realizations for degenerate Poisson brackets can be traced back to Lie, who used the name “function group” [18]. Lie proved that such a realization always exists locally for any Poisson manifold of constant rank. The local existence theorem of symplectic realizations for general Poisson manifolds was proved by Weinstein in 1983 [32]. The proof was highly nontrivial and used the local structure theorem for Poisson manifolds. Subsequently, in 1987, Karasev [11] and Weinstein [33] proved independently the existence of a global symplectic realization for any Poisson manifold. In fact, they found that by a suitable choice, such a realization admits **automatically** a local groupoid structure which is compatible with the symplectic structure in a certain sense. The global form of this notion is what is now called a symplectic groupoid. Symplectic groupoids have their own origin in quantization theory [13]. However,

*1991 *Mathematics Subject Classification*. Primary 58F05. Secondary 17B66, 22A22, 58H05.

†Research partially supported by NSF Grant DMS95-04913.

it has been quite mysterious why the groupoid structure and symplectic structure enter into the picture of a Poisson manifold in such a compatible and striking manner.

On the other hand, Poisson groups have been intensively studied as a classical limit of quantum groups. The theory of Poisson groups established a precise relation between Poisson structures on the groups and their infinitesimal invariants, Lie bialgebras. In order to understand symplectic groupoids using the techniques of Poisson group theory and to unify both theories in a general framework, Weinstein in 1988 introduced the notion of Poisson groupoid [34]. Lie bialgebroids were introduced and studied by the present authors [26] in 1994 as the infinitesimal invariants of Poisson groupoids: given a Poisson groupoid G , the Lie algebroid of the underlying Lie groupoid, together with the Lie algebroid structure on the dual A^*G [34], form a Lie bialgebroid. Lie bialgebroids are found to be connected with various subjects in Poisson geometry ranging from Poisson-Nijenhuis structures to Dirac structures; see, for example, [14], [15], [20], [19], [25]. However, it has remained an unsettled problem whether an arbitrary Lie bialgebroid can be integrated to a Poisson groupoid.

In this paper, we give an affirmative answer to this question. We prove that a Lie bialgebroid structure on the Lie algebroid of a (suitably simply-connected) Lie groupoid can be integrated to give a Poisson groupoid structure on the underlying groupoid. This result extends the well-known result that a Lie bialgebra (of finite dimension over \mathbb{R} or \mathbb{C}) is the Lie bialgebra of a Poisson group [9], [22]. At the other extreme, it also shows that if a Poisson manifold P has a cotangent Lie algebroid which integrates to a Lie groupoid $G \rightrightarrows P$, then G has a symplectic groupoid structure integrating the Poisson structure of P . This is a large part of the local integrability of Poisson manifolds [12], [32], [33]. In particular, we obtain as a consequence a new proof of the existence of local symplectic groupoids for general Poisson manifolds.

Within this general framework, the geometric origin of the symplectic and groupoid structures on a symplectic groupoid becomes transparent. Given a Poisson manifold P , its cotangent bundle T^*P carries a Lie algebroid structure (see Section 5 for the precise definition). Now if this integrates to a Lie groupoid Γ (assumed α -simply connected), the canonical Lie algebroid structure on the dual of T^*P , namely the tangent bundle TP , induces a Poisson structure on Γ , which in this case is symplectic. The compatibility condition between the two Lie algebroid structures then ensures the compatibility condition between the groupoid and symplectic structures, and so Γ is a symplectic groupoid.

The integrability of Lie bialgebras reduces essentially to the lifting, or integration, of Lie algebra 1-cocycles, but for general Lie bialgebroids this approach is not available: there is no satisfactory adjoint representation for a general Lie algebroid, and if one were to treat the Lie algebroids as infinite dimensional Lie algebras, then the problems would be at least as great as those resolved here.

In Theorem 6.2 of [26], we proved that a Lie algebroid A , whose dual A^* also has a Lie algebroid structure, is a Lie bialgebroid if and only if a certain map $\Pi: T^*A^* \rightarrow TA$ is a Lie algebroid morphism. Here T^*A^* is the cotangent Lie algebroid for the Poisson structure on A^* induced by A , and TA is the tangent Lie algebroid structure on base TP , of $A \rightarrow P$ [26, §5]. This result now allows us, after some work, to reduce the integrability

problem to the integration of Lie algebroid morphisms. This approach is not, in fact, so very far from the integration of cocycles—the standard proof of the integrability of cocycles proceeds by treating them as morphisms into semi-direct products. Here, however, none of the Lie algebroid structures need be a semi-direct product.

Unlike the case of Lie algebras, a Lie algebroid need not arise from a Lie groupoid. If G is a Poisson groupoid and the Lie algebroid dual A^*G integrates to an α -simply connected groupoid G^* , then the results of the present paper make G^* into a Poisson groupoid also. The further relations between a Poisson groupoid and its dual will be investigated elsewhere.

The results of the present paper settle, we believe, any remaining doubt that the concept of Lie bialgebroid is the correct infinitesimal form of the concept of Poisson groupoid. This is an important point, in view of the complexity of the work on notions of double for Lie bialgebroids [20], [25].

We begin in §2 by giving some preliminary material concerning affine multivector fields on Lie groupoids. In §3 we recall the basic definitions and main results from [26]. The main integrability theorem is proved in §4. In §5 we consider the case of symplectic groupoids proving, in particular, that if the cotangent Lie algebroid of a Poisson manifold integrates to an α -simply connected groupoid, then the groupoid has a natural structure of symplectic groupoid. Thus the existence of a local symplectic groupoid for a Poisson manifold follows as a consequence. Finally in §6 we give a full proof of the integrability result for Lie algebroid morphisms on which the main results depend.

We have minimized the repetition of material from [26], and so have used the same notation conventions.

We are once again very grateful to Alan Weinstein for conversations over an extended period on the material of this paper. We also thank the Isaac Newton Institute at Cambridge and the organizers of the workshop on Symplectic Geometry. The second author wishes to thank IHES and Max-Planck-Institut for their hospitality while part of the work was being done.

2 Affine multivector fields

Throughout this section, we fix a Lie groupoid $G \rightrightarrows M$ and denote its Lie algebroid by AG . We follow the conventions of [26]; in particular, for $g, h \in G$ the product gh is defined if $\alpha g = \beta h$.

We recall the exponential map for a Lie groupoid [23], [6]. Given $X \in \Gamma AG$, the flows φ_t of the corresponding right invariant vector field \overrightarrow{X} are left translations. Assume for convenience that φ_t is global and define $\exp tX: M \rightarrow G$ by $\exp tX(m) = \varphi_t(1_m)$. Then $\exp tX$ is a section of $\alpha: G \rightarrow M$, and $\beta \circ \exp tX: M \rightarrow M$ is the flow of $a(X) \in \mathcal{X}(M)$. Call any section $\mathcal{K}: M \rightarrow G$ of α for which $\beta \circ \mathcal{K}$ is a diffeomorphism a *bisection* of G (*admissible section* in [23]). Then $L_{\mathcal{K}}(g) = \mathcal{K}(\beta g)g$ is the left translation corresponding to \mathcal{K} , and $R_{\mathcal{K}}(g) = g\mathcal{K}((\beta \circ \mathcal{K})^{-1}(\alpha g))$ is the right translation. We often denote the tangents of $L_{\mathcal{K}}$ and $R_{\mathcal{K}}$ by the same symbols. By $Ad_{\mathcal{K}}$ we denote the groupoid automorphism $L_{\mathcal{K}} \circ R_{\mathcal{K}}^{-1}$. It is clear that $Ad_{\mathcal{K}}$ leaves M invariant, and its restriction to M is the map $\beta \circ \mathcal{K}$. The set

of all (global) bisections $\mathcal{G}(G)$ forms a group under $\mathcal{K}_1 \circ \mathcal{K}_2(m) = \mathcal{K}_1(\text{Ad}_{\mathcal{K}_2}(m))\mathcal{K}_2(m)$. In general $\exp tX$ is a local bisection (in an evident sense), and is only defined for small t .

Alternatively, one may identify a bisection with its image, in which case a bisection is a submanifold of G for which the restrictions of both α and β are diffeomorphisms [6]. Then $\exp tX$, for $X \in \Gamma AG$, is the submanifold of G obtained by flowing the identity space M under the flow φ_t of \overrightarrow{X} . We will use both points of view in what follows. The following formulas are frequently used in the paper.

$$\alpha(\exp tX(m)) = m, \quad \beta(\exp tX(m)) = \text{Ad}_{\exp tX}(m).$$

Definition 2.1 *A multivector field D on G is affine if for any $x, y \in G$ such that $\alpha(x) = \beta(y) = m$ and any bisections \mathcal{X}, \mathcal{Y} through the points x, y , we have*

$$D(xy) = R_y D(x) + L_x D(y) - R_y L_x D(1_m). \quad (1)$$

To appreciate this definition, recall that TG inherits a groupoid structure on base TM from $G \rightrightarrows M$ with source map $T(\alpha)$, target map $T(\beta)$, and composition $X \bullet Y = T(\kappa)(X, Y)$, where κ is the composition in G . One of us proved [36, 2.4] that if $X \in T_x(G)$ and $Y \in T_y(G)$ have $T(\alpha)(X) = T(\beta)(Y) = W$, then this composition may also be given by

$$X \bullet Y = T(L_{\mathcal{X}})(Y) + T(R_{\mathcal{Y}})(X) - T(L_{\mathcal{X}})T(R_{\mathcal{Y}})(T(1)(W)) \quad (2)$$

where \mathcal{X}, \mathcal{Y} are any (local) bisections of G with $\mathcal{X}(\alpha x) = x$ and $\mathcal{Y}(\alpha y) = y$.

For the case of affine multivector fields on groups, see [35, §4]. Affine multivector fields arise in many natural ways. For instance, for any $K \in \Gamma(\wedge^k AG)$, the multivector field $\overrightarrow{K} - \overleftarrow{K}$, the difference between the right and left translations of K , is an affine multivector field on G . We deal elsewhere [27] with the special features of affine vector fields.

The following theorem gives a very useful characterization of affine multivector fields. See [22] and [35] for the case of groups.

Theorem 2.2 *Let G be α -connected. For a multivector field D on G , the following statements are equivalent:*

- (i). D is affine;
- (ii). For any right (left) invariant vector field \overrightarrow{X} (respectively \overleftarrow{X}), the Lie derivative $L_{\overrightarrow{X}} D$ ($L_{\overleftarrow{X}} D$) is right (left) invariant;
- (iii). for any left invariant vector field \overleftarrow{X} and right invariant vector field \overrightarrow{Y} , $L_{\overrightarrow{Y}} L_{\overleftarrow{X}} D = 0$.

Proof. The equivalence between (ii) and (iii) is quite evident.

(i) \iff (ii) Our proof here follows that of Theorem 3.1 in [36].

Suppose that D is an affine multivector field. It suffices to show that statement (ii) holds for any compactly supported $X \in \Gamma_c(AG)$, since the evaluation of $L_{\overrightarrow{X}}D$ at any particular point only depends on the local germ of X . Therefore, without loss of generality, we can assume that \overrightarrow{X} is a complete vector field on G . Let $\mathcal{X}_t = \exp tX \in \mathcal{G}(G)$, the one parameter family of bisections generated by $X \in \Gamma_c(AG)$. Suppose that \mathcal{Y} is any bisection through a point $y \in G$ with $\beta(y) = m$. Let $x_t = \mathcal{X}_t(m)$, the flow generated by \overrightarrow{X} with initial point 1_m . According to Equation (1), we have

$$D(x_ty) = R_{\mathcal{Y}}D(x_t) + L_{\mathcal{X}_t}D(y) - L_{\mathcal{X}_t}R_{\mathcal{Y}}D(1_m),$$

which is equivalent to

$$L_{\mathcal{X}_t^{-1}}D(x_ty) = (R_{\mathcal{Y}} \circ L_{\mathcal{X}_t^{-1}})D(x_t) + D(y) - R_{\mathcal{Y}}D(1_m). \quad (3)$$

Here, both sides of Equation (3) are elements in $\wedge^k T_x G$, where k is the degree of D . Taking the derivative at $t = 0$, one obtains that

$$(L_{\overrightarrow{X}}D)(y) = R_{\mathcal{Y}}[(L_{\overrightarrow{X}}D)(1_m)]. \quad (4)$$

This implies that $L_{\overrightarrow{X}}D$ is right-invariant, according to Lemma 3.2 in [36].

Conversely, integrating Equation (4), one gets immediately that

$$L_{\mathcal{X}_t^{-1}}D(x_ty) - D(y) = R_{\mathcal{Y}}[L_{\mathcal{X}_t^{-1}}D(x_t) - D(1_m)]. \quad (5)$$

Thus Equation (1) follows if $\mathcal{Y} = \exp X$ for any $X \in \Gamma_c(AG)$. In the general case it follows by applying Equation (5) repeatedly and the following result.

Lemma 2.3 *For any $x \in G$ and any bisection \mathcal{K} , there exist $X_1, \dots, X_n \in \Gamma_c(AG)$ such that $\tilde{\mathcal{K}} = \exp X_1 \dots \exp X_n$ has the value x at αx and $\mathcal{K}_*|_{\alpha x} = \tilde{\mathcal{K}}_*|_{\alpha x}$.*

Proof. Let \mathcal{U} denote the set of values $\exp tX(m)$ as X ranges through $\Gamma_c(AG)$, t ranges through \mathbb{R} , and m through M . Clearly, the statement holds if $x \in \mathcal{U}$. It is easy to verify that the intersections of \mathcal{U} with each α -fibre is open and it follows by a modification of a standard argument [23, II§3] that the subgroupoid H generated by \mathcal{U} has each α -fibre open in the corresponding α -fibre of G . Since G is α -connected it follows that $H = G$.

□

Any bivector field D induces a map $D^\# : T^*G \longrightarrow TG$ defined by $\langle \omega_2, D^\#(\omega_1) \rangle = \langle D, \omega_1 \wedge \omega_2 \rangle$. It is well known that both T^*G and TG have Lie groupoid structures [6], [31], that for TG being given above (2). For T^*G we use the conventions of [26, §7]: we take the source $\tilde{\alpha}$ and target $\tilde{\beta}$ to be given by

$$\tilde{\alpha}(\omega)(X) = \omega(T(L_g)(X - T(1)(a(X)))), \quad \tilde{\beta}(\omega)(Y) = \omega(T(R_g)(Y)), \quad (6)$$

where $\omega \in T_g^*G$, $X \in A_{\alpha g}G$ and $Y \in A_{\beta g}G$. If $\theta \in T_h^*G$ and $\tilde{\alpha}(\theta) = \tilde{\beta}(\omega)$ then $\alpha h = \beta g$ and we define $\theta \bullet \omega \in T_{hg}^*G$ by

$$(\theta \bullet \omega)(Y \bullet X) = \theta(Y) + \omega(X),$$

where $Y \in T_hG$, $X \in T_gG$. The identity element $\tilde{1}_\varphi \in T_{1_m}^*G$ corresponding to $\varphi \in A_m^*G$ is defined by $\tilde{1}_\varphi(T(1)(x) + X) = \varphi(X)$ for $X \in A_mG$, $x \in T_m(M)$.

We can now give the following criterion for D to be affine, which will be important in the proof of Theorem 4.1. An immediate consequence of this criterion is that the Poisson tensor on a Poisson groupoid is affine, one of the main results proved in [36]. In fact, our proof here is essentially borrowed from that in [36].

Proposition 2.4 *Let D be a bivector field on G . If*

$$\begin{array}{ccc} T^*G & \xrightarrow{D^\#} & TG \\ \Downarrow & & \Downarrow \\ A^*G & \xrightarrow{a_*} & TP \end{array} \quad (7)$$

is a Lie groupoid morphism, then the bivector field D is affine.

Proof. By $\Lambda \subset G \times G \times G$, we denote the graph of groupoid multiplication, and by Ω , we denote the subset of $G \times G \times G \times G$ consisting of all elements (z, y, x, w) such that $w = yz^{-1}x$. Weinstein [35] calls Ω the *affinoid diagram* corresponding to the groupoid G . The graph of multiplication of the groupoid $T^*G \rightrightarrows A^*G$ is $\bar{N}^*\Lambda$, which is the subset of $T^*(G \times G \times G)$ obtained from the conormal bundle $N^*\Lambda$ by multiplying the cotangent vectors in the last factor by -1 . Thus, by assumption, $D^\# \bar{N}^*\Lambda \subset T\Lambda$. Following the proof of Theorem 4.5 in [35], we can show that $D^\# N^*\Omega \subset T\Omega$. For any $x, y \in G$ such that $\alpha(x) = \beta(y) = m$, it is clear that $(z, y, x, 1_m)$, with $z = xy$, is an element of Ω . For any $\xi \in T_zG$, it follows from Lemma 2.6 in [36] that $(-\xi, L_\chi^*\xi, R_y^*\xi, -L_\chi^*R_y^*\xi)$ is a conormal vector to Ω . Therefore, for any $\xi, \eta \in T_zG$, we have

$$-D(z)(\xi, \eta) + D(y)(L_\chi^*\xi, L_\chi^*\eta) + D(x)(R_y^*\xi, R_y^*\eta) - D(w)(L_\chi^*R_y^*\xi, L_\chi^*R_y^*\eta) = 0.$$

This implies Equation (1) immediately. □

A bivector field D for which $D^\#$ is a morphism might be called *multiplicative*. Not all affine bivector fields are multiplicative.

Proposition 2.5 *The Schouten bracket of affine multivector fields is still affine.*

Proof. Suppose that D_1 and D_2 are affine multivector fields on G . For any $X, Y \in \Gamma(AG)$, we have

$$L_Y \overleftarrow{L}_X [D_1, D_2] = [L_Y \overleftarrow{L}_X D_1, D_2] + [L_X \overleftarrow{L}_Y D_1, L_Y \overleftarrow{L}_X D_2] + [L_Y \overleftarrow{L}_X D_1, L_X \overleftarrow{L}_Y D_2] + [D_1, L_Y \overleftarrow{L}_X D_2].$$

Each summand on the right hand side is easily seen to be zero according to Theorem 2.2, so it follows that $[D_1, D_2]$ is affine. □

As in the case of groups, the derivative of an affine k -vector field D can be introduced, and is a map $dD : \Gamma(AG) \longrightarrow \Gamma(\wedge^k AG)$ defined as follows. For any $X \in \Gamma(AG)$, $dD(X)$ is defined to be the element in $\Gamma(\wedge^k AG)$ whose right translation is $L_X \overleftarrow{D}$. It is easy to see that if $\Gamma(AG)$ is considered as an infinite dimensional Lie algebra with $\Gamma(\wedge^k AG)$ considered as a $\Gamma(AG)$ -module in a natural way, then dD may be considered a Lie algebra 1-cocycle. It is not clear in general whether such a Lie algebra 1-cocycle can be lifted to an affine multivector field. However, the following theorem indicates that if it exists, then it is unique.

Theorem 2.6 *Let D be an affine multivector field on an α -connected Lie groupoid G . Then D is zero if and only if D vanishes on the unit space M and $dD = 0$.*

Proof. Let $X \in \Gamma_c(AG)$ be any compactly supported section and let $\mathcal{X}_t = \exp tX \in \mathcal{G}(G)$ be the one parameter family of bisections generated by X . Fixing any $m \in M$, let $x_t = \exp tX(m)$, and let $f_m(t)$ be the vector in $\wedge^k T_m G$ given by $f_m(t) = L_{\mathcal{X}_t^{-1}} D(x_t)$, where k is the degree of D . For any $t, s \in \mathbb{R}$,

$$\begin{aligned} f_m(t+s) &= L_{\mathcal{X}_{t+s}^{-1}} D(x_{t+s}) \\ &= L_{\mathcal{X}_{t+s}^{-1}} D[\exp sX(v) \cdot \exp tX(m)] \\ &= L_{\mathcal{X}_{t+s}^{-1}} [R_{\mathcal{X}_t} D(\exp sX(v)) + L_{\mathcal{X}_s} D(\exp tX(m))] \\ &= Ad_{\mathcal{X}_t^{-1}} L_{\mathcal{X}_s^{-1}} [D(\exp sX(v))] + L_{\mathcal{X}_t^{-1}} D(\exp tX(m)) \\ &= Ad_{\mathcal{X}_t^{-1}} f_v(s) + f_m(t), \end{aligned}$$

where $v = \beta[\exp tX(m)] = Ad_{\mathcal{X}_t} m$, and in the third equality we have used Equation (1).

By taking the derivative with respect to s at 0, it follows immediately that

$$\frac{d}{dt} f_m(t) = (Ad_{\mathcal{X}_t}^{-1})_* (L_X \overleftarrow{D})(v) = 0.$$

Therefore, $f_m(t) = 0$ for all t since $f_m(0) = 0$. This shows that $D(\exp tX(m)) = 0$ for all t . Since any element in G can be written as a product of elements of the form $\exp X(m)$, it thus follows that D is identically zero on G , again by Equation (1). □

3 Poisson groupoids and Lie bialgebroids

In this section we briefly review some material from [26].

The concept of Lie bialgebroid can be defined in terms of a cocycle-type condition, using a generalized Schouten calculus, but in this paper we will be mainly concerned with the following equivalent characterization in terms of morphisms of Lie algebroid structures.

Definition 3.1 ([26, 3.1]) *Suppose that $A \rightarrow P$ is a Lie algebroid, and that its dual bundle $A^* \rightarrow P$ also carries a Lie algebroid structure. Then (A, A^*) is a Lie bialgebroid if for any $X, Y \in \Gamma(A)$,*

$$d_*[X, Y] = L_X d_* Y - L_Y d_* X. \quad (8)$$

For an alternative treatment of this definition, see Kosmann–Schwarzbach [14].

Theorem 3.2 ([26, 6.2]) *Suppose that $q: A \rightarrow P$ is a Lie algebroid such that its dual vector bundle $q_*: A^* \rightarrow P$ also has a Lie algebroid structure. Let a, a_* be their anchors. Then (A, A^*) is a Lie bialgebroid if and only if*

$$\begin{array}{ccc} T^*(A^*) & \xrightarrow{\quad \Pi \quad} & TA \\ \downarrow & & \downarrow \\ A^* & \xrightarrow{\quad a_* \quad} & TP \end{array} \quad (9)$$

is a Lie algebroid morphism, where the domain $T^(A^*) \rightarrow A^*$ is the cotangent Lie algebroid induced by the Poisson structure on A^* , the target $TA \rightarrow TP$ is the tangent prolongation of A , and $\Pi: T^*(A^*) \rightarrow TA$ is the composition of the isomorphism $R: T^*A^* \rightarrow T^*A$ described below with $\pi_A^\# : T^*A \rightarrow TA$.*

We recall the structures used in this theorem. Given any vector bundle $q: A \rightarrow P$, the map $T(q): TA \rightarrow TP$ has a vector bundle structure obtained by applying the tangent functor to the operations in $A \rightarrow P$. The operations in $TA \rightarrow TP$ are consequently vector bundle morphisms with respect to the tangent bundle structures in $TA \rightarrow A$ and $TP \rightarrow P$ and so TA with these two structures is a double vector bundle which we call the *tangent double vector bundle of $A \rightarrow P$* (see [24, §1] and references given there). If $q: A \rightarrow P$ is a Lie algebroid then there is a Lie algebroid structure on $T(q): TA \rightarrow TP$ defined in [26, 5.1] with respect to which $p_A: TA \rightarrow A$ is a Lie algebroid morphism over $p_P: TP \rightarrow P$; we now call this the *tangent prolongation of $A \rightarrow P$* .

For a general vector bundle $q: A \longrightarrow P$, there is also a double vector bundle

$$\begin{array}{ccc}
 T^*A & \xrightarrow{r_A} & A^* \\
 c_A \downarrow & & \downarrow q_* \\
 A & \xrightarrow{q} & P
 \end{array} \tag{10}$$

where c_A is the usual cotangent bundle. For the structure on r_A see [26, p.430]. Elements of T^*A can be represented locally as (ω, X, φ) where $\omega \in T_u^*P$, $X \in A_u$, $\varphi \in A_u^*$ for some $u \in P$. In these terms a canonical map $R: T^*A^* \longrightarrow T^*A$ can be defined by $R(\omega, \varphi, X) = (-\omega, X, \varphi)$; for an intrinsic definition see [26, 5.5]. This R is an isomorphism of double vector bundles preserving the side bundles; that is to say, it is a vector bundle morphism over both A and A^* .

If A^* has a Lie algebroid structure then the dual Poisson structure π_A on A has associated map $\pi_A^\#: T^*A \longrightarrow TA$ which is a morphism of double vector bundles over $a_*: A^* \longrightarrow TP$ and id_A . In a Lie bialgebroid (A, A^*) the same is consequently true of Π .

A *Poisson groupoid* (Weinstein [34]) is a Lie groupoid $G \rightrightarrows P$ together with a Poisson structure π_G on G such that the graph of the groupoid multiplication $\Lambda = \{(h, g, hg) \mid \alpha h = \beta g\}$ is a coisotropic submanifold of $G \times G \times \overline{G}$. Any Poisson manifold P gives rise to a Poisson groupoid $\overline{P} \times P \rightrightarrows P$ where \overline{P} is P with the opposite structure and the groupoid structure is $(w, v)(v, u) = (w, u)$. Any Poisson Lie group is of course a Poisson groupoid. We consider examples further below.

It was shown in [34] that the manifold of identity elements of a Poisson groupoid G is coisotropic in G , and its conormal bundle $N^*(P)$ thereby acquires a Lie algebroid structure. This conormal bundle may be identified with A^*G , the dual vector bundle of AG , in a standard way, and we will always take A^*G with this Lie algebroid structure. Denote the anchor of A^*G by a_* .

In this paper we will use the following equivalent condition [2], [26, 8.1] repeatedly.

Proposition 3.3 *Let $G \rightrightarrows P$ be a Lie groupoid with a Poisson structure π_G . Then G is a Poisson groupoid with respect to π_G if and only if $\pi_G^\#: T^*G \rightarrow TG$ is a morphism of Lie groupoids over some map $a_*: A^*G \rightarrow TP$ (which is then the anchor of the Lie algebroid dual).*

In a Poisson groupoid $G \rightrightarrows P$ we can therefore apply the Lie functor to $\pi_G^\#$ and obtain a morphism of Lie algebroids $A(\pi_G^\#): AT^*G \rightarrow ATG$.

For any Lie groupoid $G \rightrightarrows M$, the tangent bundle projection $p_G: TG \rightarrow G$ is a groupoid morphism over $p_M: TM \rightarrow M$ and applying the Lie functor gives a canonical morphism $A(p_G): ATG \rightarrow AG$. This acquires a vector bundle structure by applying A to the operations

in $TG \longrightarrow G$. This yields a system of vector bundles

$$\begin{array}{ccc}
ATG & \xrightarrow{q_{TG}} & TM \\
A(p_G) \downarrow & & \downarrow p_M \\
AG & \xrightarrow{q_G} & M
\end{array} \tag{11}$$

in which ATG has two vector bundle structures, the maps defining each being morphisms with respect to the other; that is to say, ATG is a double vector bundle.

Associated with the vector bundle $q_G: AG \longrightarrow M$ is the tangent double vector bundle

$$\begin{array}{ccc}
TAG & \xrightarrow{T(q_G)} & TM \\
p_{AG} \downarrow & & \downarrow p_M \\
AG & \xrightarrow{q_G} & M.
\end{array} \tag{12}$$

It is shown in [26, 7.1] that there is a canonical map

$$j_G: TAG \rightarrow ATG$$

which is an isomorphism of double vector bundles preserving the side bundles. This j_G is a restriction of the canonical involution on T^2G .

Similarly, the cotangent groupoid structure $T^*G \rightrightarrows A^*G$ is defined by maps which are vector bundle morphisms and, reciprocally, the operations in the vector bundle $c_G: T^*G \longrightarrow G$ are groupoid morphisms. Taking the Lie algebroid of $T^*G \rightrightarrows A^*G$ we get a double vector bundle

$$\begin{array}{ccc}
AT^*G & \xrightarrow{q_{T^*G}} & A^*G \\
A(c_G) \downarrow & & \downarrow q_* \\
AG & \xrightarrow{q_G} & M,
\end{array} \tag{13}$$

where the vector bundle operations in $AT^*G \longrightarrow AG$ are obtained by applying the Lie functor to those in $T^*G \longrightarrow G$.

It follows from the definitions of the operations in $T^*G \rightrightarrows A^*G$ that the canonical pairing $F: T^*G *_G TG \longrightarrow \mathbb{R}$, $(\omega, X) \mapsto \langle \omega, X \rangle$, can be considered a groupoid morphism into the additive group(oid) \mathbb{R} . Here

$$T^*G *_G TG = \{(\omega, X) \mid \omega \in T^*G, X \in TG \text{ such that } c_G(\omega) = p_G(X)\}$$

is the pullback groupoid of c_G and p_G ; it has base $A^*G *_M TM$. Hence F induces a Lie algebroid morphism $A(F): AT^*G *_M AG \rightarrow \mathbb{R}$, where $AT^*G *_M AG$ is the pullback Lie algebroid [10, §1]. As noted in [26, 7.2], $A(F)$ is nondegenerate, and so induces an isomorphism of double vector bundles $i_G: AT^*G \rightarrow A^\bullet TG$, where $A^\bullet TG$ is the dual of $ATG \rightarrow AG$. Now dualizing $j_G: TAG \rightarrow ATG$ over AG , we define

$$j'_G = j_G^* \circ i_G: AT^*G \rightarrow T^*AG;$$

this is an isomorphism of double vector bundles preserving the side bundles. The following result is now immediate.

Proposition 3.4 *For $\nu \in AT^*G$ and $\xi \in ATG$ with $A(c_G)(\nu) = A(p_G)(\xi)$ we have*

$$A(F)(\nu, \xi) = \langle j'_G(\nu), j_G^{-1}(\xi) \rangle,$$

where the pairing on the right is the canonical pairing between T^*AG and TAG .

It is proved in [26, 7.3] that the composition $(j'_G)^{-1} \circ R$ is equal to the canonical isomorphism $s: T^*A^*G \rightarrow AT^*G$ arising from the symplectic groupoid structure on T^*G , that is,

$$s = (j'_G)^{-1} \circ R. \quad (14)$$

In particular, s is an isomorphism of double vector bundles from

$$\begin{array}{ccc} T^*A^*G & \xrightarrow{c_{A^*G}} & A^*G \\ r_{A^*G} \downarrow & & \downarrow q_* \\ AG & \xrightarrow{q_G} & M \end{array} \quad (15)$$

to (13) which preserves the side bundles.

Returning to the Poisson groupoid $G \rightrightarrows P$, we now have the morphism

$$j_G^{-1} \circ A(\pi_G^\#) \circ j'_G{}^{-1}: T^*AG \rightarrow TAG$$

and it is proved in [26, §8] that this is equal to $\pi_{AG}^\#$, the Poisson tensor map for the Poisson structure on AG dual to the Lie algebroid structure on A^*G . It follows that $\pi_{AG}^\# \circ R = j_G^{-1} \circ A(\pi_G^\#) \circ s$ is a morphism of Lie algebroids over a_* , and hence that (AG, A^*G) is a Lie bialgebroid. (An alternative proof of this result is given in [36, 3.5].) The integrability proof of §4 will consist essentially of reversing these steps.

Finally in this section we say something about examples. One might expect that the standard algebraic constructions for Lie algebroids [10] would have Lie bialgebroid analogues which would provide a rich source of examples. By and large, however, this appears not to be the case.

Example 3.5 ([21]) Let $A \rightarrow M$ be a Lie algebroid and let Λ be an element of $\Gamma(\wedge^2 A)$ with $\Lambda^\#: A^* \rightarrow A$ defined by $\langle \varphi, \Lambda^\#(\psi) \rangle = \Lambda(\psi, \varphi)$ for $\varphi, \psi \in \Gamma A^*$. Then [21, §2], defining a bracket on ΓA^* by

$$[\varphi, \psi] = L_{\Lambda^\#(\varphi)}(\psi) - L_{\Lambda^\#(\psi)}(\varphi) - d(\Lambda(\varphi, \psi))$$

and defining $a_* = a \circ \Lambda^\#$, makes A^* a Lie algebroid if and only if $a_*[\Lambda, \Lambda]^\# = 0$ and $L_X([\Lambda, \Lambda]) = 0$ for all $X \in \Gamma A$; when these conditions hold, it further follows that (A, A^*) is a Lie bialgebroid, called *exact*. It is proved in [21] that when A is the Lie algebroid of a Lie groupoid G , an exact Lie bialgebroid structure integrates to a Poisson groupoid structure on G ; if Λ is the r -matrix for AG , then the Poisson structure on G is given by $\pi = \overrightarrow{\Lambda} - \overleftarrow{\Lambda}$. In particular, twisted Poisson groupoid structures can be defined on the trivial Lie groupoid $\overline{P} \times G \times P$, where (P, π_P) is a Poisson manifold and G is a Lie group, by defining $\Lambda^\#: T^*P \oplus (P \times \mathfrak{g}^*) \rightarrow TP \oplus (P \times \mathfrak{g})$ by $\Lambda = \pi_P + X \wedge \xi + r$, where $X \in \mathcal{X}(P)$ and $r: P \rightarrow \wedge^2 \mathfrak{g}$ satisfy $L_X(\pi_P) = 0$ and $[\xi, r] = 0$, and $[r, r]$ is pointwise ad-invariant.

Example 3.6 ([15]) A Nijenhuis structure on a manifold M is an endomorphism $N: TM \rightarrow TM$ whose Nijenhuis torsion is zero. If N is a Nijenhuis structure, then a deformed Lie algebroid structure may be defined on TM with N as anchor and bracket $[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]$. Denote this Lie algebroid by $(TM)_N$.

If M also has a Poisson structure π , then N and π are called *compatible* if $N\pi = \pi N^*$ and a certain torsion-like expression in N and π vanishes; M with π and N is then called a *Poisson–Nijenhuis structure* [16]. When this is the case, T^*M with the usual cotangent Lie algebroid structure forms a Lie bialgebroid with $(TM)_N$. Indeed compatibility of π and N is equivalent to T^*M and $(TM)_N$ forming a Lie bialgebroid.

Example 3.7 ([25]) Let S be any double Lie groupoid with side groupoids $H \rightrightarrows M$ and $V \rightrightarrows M$ and core groupoid $C \rightrightarrows M$ [4], [24]. The horizontal groupoid structure $S \rightrightarrows V$ yields a Lie algebroid $A_H S \rightarrow V$ which is also a groupoid over AH ; the structure maps of this groupoid structure are Lie algebroid morphisms, and the resulting double structure has core Lie algebroid $AC \rightarrow M$. It follows therefore from the duality of Pradines that $A_H^* S$ has a groupoid structure on base $A^* C$. With respect to the dual Poisson structure, this is a Poisson groupoid, inducing on $A^* C$ the dual Poisson structure. Performing the same construction with the vertical structure yields a Poisson groupoid structure on $A_V^* S \rightrightarrows A^* C$, which is dual to $A_H^* S$.

In a similar way any \mathcal{LA} -groupoid $(\Omega; A, V; M)$ with core Lie algebroid $K \rightarrow M$ gives rise to a Poisson groupoid $\Omega^* \rightrightarrows K^*$.

4 Integration of Lie bialgebroids

This section is devoted to the proof of the following theorem, which is the main result of the paper.

Theorem 4.1 *Let (AG, A^*G) be a Lie bialgebroid where AG is the Lie algebroid of an α -simply connected Lie groupoid $G \rightrightarrows P$. Then there is a unique Poisson structure on G that makes G into a Poisson groupoid with Lie bialgebroid (AG, A^*G) .*

The hypotheses of the Theorem are fixed throughout the remainder of the section.

By assumption,

$$\Pi = \pi_{AG}^\# \circ R: T^*A^*G \longrightarrow TAG \quad (16)$$

is a morphism of Lie algebroids over a_* , and so

$$j_G \circ \Pi \circ s^{-1} = j_G \circ \pi_{AG}^\# \circ j'_G: AT^*G \longrightarrow ATG \quad (17)$$

is also. Since G is α -simply connected, the cotangent groupoid is also α -simply connected, and so by Theorem 6.1, the morphism integrates uniquely to a global Lie groupoid morphism

$$\begin{array}{ccc} T^*G & \xrightarrow{\pi^\#} & TG \\ \Downarrow & & \Downarrow \\ A^*G & \xrightarrow{a_*} & TP \end{array} \quad (18)$$

Therefore,

$$A(\pi^\#) = j_G \circ \pi_{AG}^\# \circ j'_G. \quad (19)$$

Lemma 4.2 *The map $\pi^\#$ commutes with the bundle projections, $p_G \circ \pi^\# = c_G$.*

Proof. Each of j_G , $\Pi = \pi_{AG}^\# \circ R$ and s is a morphism of double vector bundles and, in particular, is a morphism of vector bundles over AG . The bundle projections $AT^*G \longrightarrow AG$ and $ATG \longrightarrow AG$ are respectively $A(c_G)$ and $A(p_G)$ so it follows that

$$A(p_G) \circ A(\pi^\#) = A(c_G).$$

Now $p_G \circ \pi^\# = c_G$ follows from the uniqueness in Theorem 6.1.

□

Lemma 4.3 *$\pi^\#: T^*G \longrightarrow TG$ is a linear map.*

Proof. It first has to be shown that

$$\begin{array}{ccc} T^*G *_G T^*G & \xrightarrow{\pi^\# \times \pi^\#} & TG *_G TG \\ + \quad \downarrow & & \downarrow \quad + \\ T^*G & \xrightarrow{\pi^\#} & TG \end{array} \quad (20)$$

commutes, where both additions are the usual ones. Again, everything is a morphism of groupoids, so we can apply A and get the diagram

$$\begin{array}{ccc}
AT^*G *_G AT^*G & \xrightarrow{A(\pi^\#) \times A(\pi^\#)} & ATG *_G ATG \\
\downarrow A(+) & & \downarrow A(+) \\
AT^*G & \xrightarrow{A(\pi^\#)} & ATG.
\end{array} \tag{21}$$

This commutes because each of s , Π and j_G is a morphism of vector bundles over AG , and so $A(\pi^\#)$ is also. The result again follows by the uniqueness in Theorem 6.1. The scalar multiplication is handled in the same way.

□

Lemma 4.4 $\pi^\#$ is skew-symmetric.

Proof. Since $\pi^\#$ is already known to be linear by Lemma 4.3, it suffices to show that the canonical pairing $F: T^*G *_G TG \rightarrow \mathbb{R}$ vanishes on the graph \mathcal{H} of $\pi^\#$. Now \mathcal{H} is a subgroupoid of $T^*G *_G TG$, and is α -connected since T^*G is, so it suffices to show that $A(F): A(\mathcal{H}) \subseteq AT^*G *_G ATG \rightarrow \mathbb{R}$ is zero. But for any $\nu \in AT^*G$ we have, using Proposition 3.4 and Equations (19) and (14),

$$A(F)(\nu, A(\pi^\#)(\nu)) = \langle j'_G(\nu), \pi^\#_{AG} \circ j'_G(\nu) \rangle = \pi_{AG}(j'_G(\nu), j'_G(\nu)) = 0.$$

□

Combining the several lemmas above, and using Proposition 2.4, we have proved the following result.

Theorem 4.5 $\pi^\#$ defines an affine bivector field π on G .

Our next task is to prove that π is indeed a Poisson tensor. As usual, first we may introduce a bracket for functions on G by¹

$$\{F_1, F_2\} = \langle \pi^\# \delta F_1, \delta F_2 \rangle, \text{ for } F_1, F_2 \in C^\infty(G).$$

This bracket is obviously skew-symmetric. Also, a bracket between one-forms can be similarly introduced by

$$\{\theta_1, \theta_2\} = \delta(\pi(\theta_1, \theta_2)) - (\pi^\# \theta_2) \lrcorner \delta \theta_1 + (\pi^\# \theta_1) \lrcorner \delta \theta_2. \tag{22}$$

¹Here, as in [26], δ denotes the usual exterior differential while the symbol d is reserved for the Lie algebroid differential.

Lemma 4.6 *Let $\{\cdot, \cdot\}_P$ denote the Poisson bracket on P induced from the Lie bialgebroid (AG, A^*G) with $\pi_P^\# = a \circ a_*^* = -a_* \circ a^*$ [26, 3.6]. Then for any $f_1, f_2 \in C^\infty(P)$,*

$$\{\alpha^* f_1, \alpha^* f_2\} = \alpha^* \{f_1, f_2\}_P, \quad \{\beta^* f_1, \beta^* f_2\} = -\beta^* \{f_1, f_2\}_P.$$

Proof. Since $\pi^\#$ is a morphism of groupoids, we have $T\alpha \circ \pi^\# = a_* \circ \tilde{\alpha}$. Hence,

$$\begin{aligned} \{\alpha^* f_1, \alpha^* f_2\} &= \langle (T\alpha \circ \pi^\#)(\alpha^* \delta f_1), \delta f_2 \rangle \\ &= \langle a_* \circ \tilde{\alpha}(\alpha^* \delta f_1), \delta f_2 \rangle \\ &= \langle a_*[-a^* \delta f_1], \delta f_2 \rangle \\ &= \langle \pi_P^\# \delta f_1, \delta f_2 \rangle \\ &= \{f_1, f_2\}_P, \end{aligned}$$

where in the third equality, we have used the fact that $\tilde{\alpha}(\alpha^* \delta f_1) = -a^* \delta f_1$, which follows directly from the definition (6). The other identity can be proved similarly. \square

Since π is affine, $L_{\overrightarrow{X}}\pi$ is right invariant for any right invariant vector field \overrightarrow{X} by Theorem 2.2. The following proposition explicitly describes this invariant bivector field (compare [36, 3.1]).

Proposition 4.7 (i). *For any $X \in \Gamma AG$, we have*

$$L_{\overrightarrow{X}}\pi = [\overrightarrow{X}, \pi] = -\overrightarrow{d_* X}, \quad (23)$$

where $\overrightarrow{d_* X}$ is the right invariant bivector field on G corresponding to $d_* X \in \Gamma(\wedge^2 AG)$.

(ii). *In general, for any $K \in \Gamma(\wedge^k AG)$, we have*

$$[\overrightarrow{K}, \pi] = -\overrightarrow{d_* K}.$$

We need a lemma to prove this proposition.

Lemma 4.8 *Let $X \in \Gamma AG$ be any section, and let ℓ_X be the corresponding fibrewise linear function on A^*G . Then the hamiltonian flow of $\tilde{\beta}^* \ell_X$ on T^*G is $(L_{\exp -tX})^*$.*

Proof. First recall that for any vector field X on any manifold M , if φ_t is a (local) flow for X and f is the function on T^*M defined by $f(\omega) = X \lrcorner \omega$, then the hamiltonian flow on T^*M generated by f is $(\varphi_t^{-1})^*$. (See Corollary 4.2.11 in [1].)

Now, from the definition (6) it follows that $(\tilde{\beta}^* \ell_X)(\omega) = \langle X, \tilde{\beta} \omega \rangle = \langle \overrightarrow{X}, \omega \rangle$, for all $\omega \in T^*G$. Since \overrightarrow{X} has flows of the form $L_{\exp tX}$, this completes the proof. \square

Proof of Proposition 4.7 Since $L_X^-(\pi)$ is known to be right invariant, it suffices to calculate $L_X^-(\pi)(\varphi_1, \varphi_2)$ for $\varphi_1, \varphi_2 \in A_m^*G$, where we identify $\varphi \in A_m^*G$ with the identity element $\tilde{1}_\varphi \in T_{1_m}^*G$ defined by $\tilde{1}_\varphi(T(1)(x) + X) = \varphi(X)$ for $x \in TP$, $X \in AG$. Let $\exp tX \in \mathcal{G}(G)$ be the family of bisections on G generated by X , and write $g_t = \exp tX(m)$. Now

$$\begin{aligned} \langle L_X^-(\pi)(1_m), \varphi_1 \wedge \varphi_2 \rangle &= \left\langle \frac{d}{dt}(L_{\exp -tX} \pi(g_t)) \Big|_0, \varphi_1 \wedge \varphi_2 \right\rangle \\ &= \frac{d}{dt} F(L_{\exp -tX}^* \varphi_2, \pi_{g_t}^\# L_{\exp -tX}^* \varphi_1) \Big|_0 \end{aligned}$$

where F is the canonical pairing. By the preceding Lemma, we have

$$\frac{d}{dt} L_{\exp -tX}^* \varphi_2 \Big|_0 = X_{\tilde{\beta}^* \ell_X} |_{\varphi_2} = s(\delta \ell_X)(\varphi_2),$$

the second equality following from the definition of s . Since $\pi^\#$ is a groupoid morphism, we also have

$$\frac{d}{dt} \pi_{g_t}^\# (L_{\exp -tX}^* \varphi_1) \Big|_0 = A(\pi^\#)(s(\delta \ell_X)(\varphi_1)).$$

Altogether, we now have

$$\langle L_X^-(\pi)(1_m), \varphi_1 \wedge \varphi_2 \rangle = A(F)(s(\delta \ell_X)(\varphi_2), A(\pi^\#)(s(\delta \ell_X)(\varphi_1))).$$

Using Proposition 3.4, this becomes

$$\begin{aligned} \langle j'_G(s(\delta \ell_X)(\varphi_2)), j_G^{-1}(A(\pi^\#)(s(\delta \ell_X)(\varphi_1))) \rangle &= \langle R(\delta \ell_X)(\varphi_2), \pi_{AG}^\#(R(\delta \ell_X)(\varphi_1)) \rangle \\ &= \pi_{AG}(R(\delta \ell_X)(\varphi_1), R(\delta \ell_X)(\varphi_2)) \\ &= -d_*(X)(\varphi_1, \varphi_2), \end{aligned}$$

the first equality following from Equations (14) and (19), and the last equality following from [26, 6.5].

Finally, (ii) is an easy consequence of (i).

□

Recall that any $\omega \in \Gamma A^*G$ can be extended by right translation to a linear form $\vec{\omega}: T^\alpha G \rightarrow \mathbb{R}$. We refer to any 1-form $\tilde{\omega}$ on G which extends $\vec{\omega}$ as an *extension* of ω .

Corollary 4.9 *Let $\tilde{\omega}, \tilde{\theta} \in \Omega^1(G)$ be any extensions of any two given sections $\omega, \theta \in \Gamma(A^*G)$, being considered as conormal vectors. Then,*

$$\{\tilde{\omega}, \tilde{\theta}\}|_P = [\omega, \theta],$$

where the bracket on the left hand side is defined by Equation (22), and that on the right hand side is the Lie algebroid bracket for A^*G .

Proof. It follows from the definition of the Lie derivative that for any $X \in \Gamma(AG)$,

$$\vec{X} \lrcorner \{\tilde{\omega}, \tilde{\theta}\} = (L_{\vec{X}}\pi)(\tilde{\omega}, \tilde{\theta}) + \pi^\# \tilde{\omega} \lrcorner \delta(\vec{X} \lrcorner \tilde{\theta}) - \pi^\# \tilde{\theta} \lrcorner \delta(\vec{X} \lrcorner \tilde{\omega}).$$

Therefore,

$$\begin{aligned} X \lrcorner \{\tilde{\omega}, \tilde{\theta}\}|_P &= (L_{\vec{X}}\pi)|_P(\omega, \theta) + (a_*\omega)(\theta X) - (a_*\theta)(\omega X) \\ &= -(d_*X)(\omega, \theta) + (a_*\omega)(\theta X) - (a_*\theta)(\omega X) \\ &= X \lrcorner [\omega, \theta], \end{aligned}$$

using the definition of the Lie algebroid coboundary d_* . This completes the proof. \square

In particular, we have

Corollary 4.10 *Let F_1, F_2 be any functions on G which are constant on P . Then $\delta\{F_1, F_2\}$ is conormal to P and its evaluation on P is equal to $[dF_1, dF_2]$, where d is the Lie algebroid coboundary for AG and the bracket is the Lie algebroid bracket on $\Gamma(A^*G)$.*

Proposition 4.11

$$[\pi, \pi]|_P = 0.$$

We need a couple of lemmas before the proof of this proposition.

Lemma 4.12 *For any $f \in C^\infty(P)$ and $H \in C^\infty(G)$,*

$$\{\beta^* f, H\} = -(\overrightarrow{a_* \delta f})(H).$$

Proof. Commencing as in the proof of Lemma 4.6,

$$\begin{aligned} \{\beta^* f, H\} &= -\langle \beta^* \delta f, \pi^\# \delta H \rangle \\ &= -\langle \delta f, T\beta \pi^\# \delta H \rangle \\ &= -\langle \delta f, a_* \tilde{\beta} \delta H \rangle \\ &= -\langle a_* \delta f, \tilde{\beta} \delta H \rangle \\ &= -\langle \overrightarrow{a_* \delta f}, \delta H \rangle \\ &= -(\overrightarrow{a_* \delta f})(H). \end{aligned}$$

where, in the second last line, we used the definition (6) of $\tilde{\beta}$. This completes the proof. \square

The following result summarizes the behaviour of $\pi^\#$ on the unit space.

Lemma 4.13 (i). For any $f \in C^\infty(P)$ and $m \in P$, $\pi_{1_m}^\#(\beta^*\delta f)$ is tangent to the α -fiber and equal to $-a_*^*\delta f$;

(ii). For any $\varphi \in A^*G$, $\pi^\#(\tilde{1}_\varphi)$ is tangent to P and equal to $a_*\varphi$.

Proof of Proposition 4.11 It suffices to show that

$$[\pi, \pi]|_P(\delta F_1, \delta F_2, \delta F_3) = \{\{F_1, F_2\}, F_3\} + c.p. \quad (24)$$

vanishes on P for any functions F_1, F_2, F_3 defined on a neighborhood of P .

Since the cotangent space $T_{1_m}^*G$ at any point $m \in P$ is spanned by the conormal space A^*G and $\beta^*T_m^*P$, it suffices to prove (24) when the F_i are either constant on P or equal to β^*f , $f \in C^\infty(P)$. We accordingly divide our proof into four different cases.

Case 1. All $F_i, i = 1, 2, 3$, are constant on P . Then $\pi^\#\delta F_1$ is tangent to P . Hence, $\{F_1, F_2\} = \langle \pi^\#\delta F_1, \delta F_2 \rangle = 0$ on P . Therefore $\{\{F_1, F_2\}, F_3\}$ vanishes on P , and so do the other two terms.

Case 2. All $F_i, i = 1, 2, 3$, are the pull backs of functions on the base space P by β . In this case, the conclusion follows from Lemma 4.6 and the Jacobi identity of the Poisson bracket on P .

Case 3. Two of the F_i are constant on P and the third one is the pull back of a function on P by β . For example, assume that F_1, F_2 are constant on P and $F_3 = \beta^*f_3$ for some $f_3 \in C^\infty(P)$. Then,

$$\{\{F_3, F_1\}, F_2\} = -(a_*\delta F_2)\{F_3, F_1\}|_P = (a_*\delta F_2)[(a_*\delta F_1)f_3].$$

Similarly, $\{\{F_2, F_3\}, F_1\} = -(a_*\delta F_1)[(a_*\delta F_2)f_3]$. On the other hand, using Corollary 4.10,

$$\{\{F_1, F_2\}, F_3\} = a_*[\delta\{F_1, F_2\}|_P](f_3) = [a_*\delta F_1, a_*\delta F_2](f_3)$$

The desired identity follows immediately.

Case 4. Assume that $F_1 = \beta^*f_1, F_2 = \beta^*f_2$ and F_3 is constant on P . Then,

$$\{\{F_2, F_3\}, F_1\} = \{\{\beta^*f_2, F_3\}, \beta^*f_1\} = \overrightarrow{(a_*^*\delta f_1)}\{\beta^*f_2, F_3\} = -\overrightarrow{(a_*^*\delta f_1)}\overrightarrow{(a_*^*\delta f_2)}(F_3),$$

where we have used Lemma 4.12 twice. Similarly, $\{\{F_3, F_1\}, F_2\} = \overrightarrow{(a_*^*\delta f_2)}\overrightarrow{(a_*^*\delta f_1)}(F_3)$.

Therefore, using [26, 3.7],

$$\{\{F_2, F_3\}, F_1\} + \{\{F_3, F_1\}, F_2\} = [\overrightarrow{(a_*^*\delta f_2)}, \overrightarrow{(a_*^*\delta f_1)}](F_3) = \overrightarrow{[a_*^*\delta\{f_2, f_1\}]}(F_3).$$

On the other hand,

$$\{\{F_1, F_2\}, F_3\} = \{\{\beta^*f_1, \beta^*f_2\}, F_3\} = -\{\beta^*\{f_1, f_2\}, F_3\} = \overrightarrow{[a_*^*\delta\{f_1, f_2\}]}(F_3).$$

Thus the RHS of Equation (24) vanishes on P . This completes the proof.

□

Proof of Theorem 4.1 Let $D = [\pi, \pi]$. Then D is an affine bivector field according to Theorem 2.2. For any $X \in \Gamma(AG)$, $L_{\overrightarrow{X}}D = 2[L_{\overrightarrow{X}}\pi, \pi] = -2[(\overrightarrow{d_*X}), \pi] = 2(\overrightarrow{d_*^2X}) = 0$ according to Proposition 4.7. Thus, $dD = 0$. Since $D|_P = 0$ by Proposition 4.11, it follows that $D = 0$ according to Theorem 2.6. That is, π is indeed a Poisson tensor. Therefore G is a Poisson groupoid according to Proposition 8.1 in [26]. Finally, Corollary 4.9 implies that the associated Lie bialgebroid is exactly isomorphic to (AG, A^*G) . The uniqueness of the Poisson structure is quite evident again by [26].

□

Note that the integrability of exact Lie bialgebroids proved in [21] and described above in Example 3.5 is more precise than Theorem 4.1 and does not require any connectedness hypotheses.

5 Symplectic groupoids

The notion of symplectic groupoid plays an important role in Poisson geometry. There already exists an extensive body of work on this subject—see, for example, [6], [8], [13]. One of the main reasons for studying Poisson groupoids and Lie bialgebroids is to understand better the relation between symplectic groupoids and Poisson groups by putting both of them into this more general framework. A particular problem is to understand how the symplectic structure of a symplectic groupoid arises. Symplectic groupoids are, of course, Poisson groupoids whose Poisson structure is symplectic. Thus one main difference between Poisson groups and general Poisson groupoids is that Poisson groups are never symplectic but the latter may be symplectic. It turns out that there is a close relation between symplectic groupoids and their base Poisson manifolds. In fact, they are in one-one correspondence in a rough local sense [6], [12], [11].

Let P be a Poisson manifold with Poisson tensor π_P . It is well-known that the cotangent bundle $T^*P \rightarrow P$ carries a natural Lie algebroid structure. Given $f \in C^\infty(P)$, denote the Hamiltonian vector field corresponding to f by X_f . Then the anchor $\pi^\#: T^*P \rightarrow TP$ is determined by $\pi^\#(f\delta g) = fX_g$. Given $\omega, \theta \in \Omega^1(P)$, and writing $X_\omega = \pi^\#\omega$ and $X_\theta = \pi^\#\theta$, the Lie algebroid bracket is

$$\{\omega, \theta\} = L_{X_\omega}\theta - L_{X_\theta}\omega - \delta(\pi(\omega, \theta)). \quad (25)$$

On the other hand, the tangent bundle TP of P has the trivial Lie algebroid structure given by the usual bracket of vector fields.

Proposition 5.1 (i). *Let P be a Poisson manifold, and T^*P the cotangent Lie algebroid described above. Then (T^*P, TP) is a Lie bialgebroid.*

(ii). *Conversely, any Lie algebroid structure on T^*P which is compatible with the trivial tangent bundle Lie algebroid TP in the sense that they become a Lie bialgebroid arises in this way.*

Proof. See Example 3.3 in [26] for (i).

Let (A, A^*) be a Lie bialgebroid such that A^* is isomorphic to TP as a Lie algebroid. It is easy to see that this isomorphism must be realized by the anchor $a_* : A^* \rightarrow TP$. According to [26, 3.6], there is a Poisson structure on P induced from the Lie bialgebroid (A, A^*) . Hence T^*P has a Lie algebroid structure as defined by Equation (25), and (T^*P, TP) becomes a Lie bialgebroid. Furthermore, the anchor $a_* : A^* \rightarrow TP$ is in fact a Lie bialgebroid morphism. Hence, its dual $a_*^* : T^*P \rightarrow A$ is a Lie algebroid morphism. Since a_* is an isomorphism, so is a_*^* . In other words, A is isomorphic to T^*P arising from a Poisson structure on P .

□

Our main theorem of the section is the following.

Theorem 5.2 *Suppose that P is a Poisson manifold and the corresponding cotangent Lie algebroid T^*P integrates to an α -simply connected groupoid G . Then G admits a natural symplectic structure which makes it into a symplectic groupoid.*

This result follows from our integration Theorem 4.1 and the following:

Theorem 5.3 *Let $G \rightrightarrows P$ be a Poisson groupoid, with Lie bialgebroid (AG, A^*G) . Let π denote the Poisson tensor on G . Then the following are equivalent:*

- (i). *(AG, A^*G) is isomorphic to the canonical Lie bialgebroid (T^*P, TP) associated to the base Poisson manifold P ;*
- (ii). *π is non-degenerate along P ;*
- (iii). *G is a symplectic groupoid.*

Proof. (i) \iff (ii). Fix any point $m \in P$. The cotangent space $T_{1_m}^*G$ is spanned by the conormal space A_m^*P and by the covectors of the form $\beta^*\delta f$ for $f \in C^\infty(P)$, so any $\omega \in T_{1_m}^*G$ can be written as $\varphi + \beta^*\delta f$ for some $\varphi \in A_m^*P$ and $f \in C^\infty(P)$. Now suppose that $\pi^\#(\varphi + \beta^*\delta f) = 0$. Then, it follows from Proposition 4.13 that $a_*\varphi - a_*^*\delta f = 0$, which implies immediately that $a_*\varphi = 0$ and $a_*^*\delta f = 0$, since $a_*\varphi$ is tangent to P while $a_*^*\delta f$ is tangent to the β -fiber. From this, we conclude that $\pi_{1_m}^\#$ is nondegenerate if and only if both $a_* : A_m^*G \rightarrow T_mP$ and $a_*^* : T_m^*P \rightarrow A_mG$ are one-to-one. This, however, is exactly equivalent to saying that $a_* : A_m^*G \rightarrow T_mP$ is one-to-one and onto. Hence, that π is non-degenerate along P is equivalent to saying that $a_* : A^*G \rightarrow TP$ is a vector bundle isomorphism, and this is also equivalent to it being a Lie bialgebroid isomorphism since it is already a Lie bialgebroid morphism. This completes our proof of the first part.

(ii) \iff (iii). We only need to prove one direction, namely, that if π is nondegenerate along P , then it is nondegenerate everywhere. Let $X \in \Gamma_c(AG)$ be any compactly supported section which is closed in the sense that $d_*X = 0$. Then, according to Corollary 3.6 in [36], $\exp X$ is a coisotropic bisection, and therefore the corresponding right translation

$R_{\exp X} : G \longrightarrow G$ is a Poisson diffeomorphism. Hence π is nondegenerate along the bisection $\exp X$. According to the first part of the proof, the Lie bialgebroid is in fact isomorphic to (T^*P, TP) . Therefore there exist abundant closed sections for $AG = T^*P$ in the sense that through any point of A there exists a closed section. Hence, through any point of G , there exists a bisection which is a product of those of the form $\exp X$ for some closed section. In other words, there always exists a coisotropic bisection through any point of G . Therefore, π is nondegenerate everywhere.

□

Remark: Theorem 5.2 fails in general if G is not assumed to be α -simply connected. An example due to Weinstein is given in [7, 5.2].

According to Pradines [30], any Lie algebroid can be integrated to a local groupoid, which can of course be assumed to be α -simply connected. As an immediate consequence of Theorem 5.2, we therefore obtain the following theorem of Karasev [11] and Weinstein [33] on the existence of local symplectic groupoids for arbitrary Poisson manifolds.

Corollary 5.4 *There always exists a local symplectic groupoid over any Poisson manifold. In particular, any Poisson manifold admits a symplectic realization.*

Although the groupoid structure is only locally defined, it includes the entire base manifold and so gives a global symplectic realization.

6 Appendix: Lifting of Lie algebroid morphisms

This section is devoted to the proof of the following theorem.

Theorem 6.1 *Let $G \rightrightarrows M$ and $H \rightrightarrows N$ be Lie groupoids. Suppose that G is α -simply connected, and that $\varphi : AG \longrightarrow AH$, $\varphi_0 : M \longrightarrow N$ is a Lie algebroid morphism. Then φ integrates uniquely to a Lie groupoid morphism $\Phi : G \longrightarrow H$.*

Proof. Let $\Delta \subseteq M \times N$ be the graph of f , and consider $G * H = (\beta_G \times \beta_H)^{-1}(\Delta)$. Also, let B be the graph of φ . Then B is a subalgebroid with base Δ of the product Lie algebroid $AG \times AH$. Identifying B with the tangent subbundle of α -fibers along the base Δ and translating it by right translations, one obtains a distribution \mathcal{D} on $G * H$, which is integrable since B is a subalgebroid.

Given any $m \in M$, let \mathcal{L}_m be the leaf of \mathcal{D} through the point $(1_m, 1_{f(m)})$. Consider the projection $p_G : \mathcal{L}_m \longrightarrow G$. It is clear, since \mathcal{D} is defined using right translations, that $p_G(\mathcal{L}_m) \subseteq G_m$, where $G_m = \alpha_G^{-1}(m) \subseteq G$. We will show that $p_G : \mathcal{L}_m \longrightarrow G_m$ is a covering map. For this, it suffices to prove that p_G possesses the path lifting property.

It is simple to see that the section space $\Gamma(AG)$ can be naturally identified with the section space $\Gamma(B)$, which in turn can be identified with the space of right invariant vector

fields on $G * H$. More explicitly, given any $X \in \Gamma(AG)$, its corresponding right invariant vector field on $G * H$, denoted by \vec{X}_B , is given by

$$\vec{X}_B(g, h) = (T(R_g)X_n, T(R_h)\varphi(X_n)), \quad \forall (g, h) \in G * H,$$

where $n = \beta_G(g)$ and $X_n \in AG|_n$. At the same time, X defines a right invariant vector field on G , which is denoted by \vec{X} . Let τ_t and ρ_t denote flows of \vec{X}_B and \vec{X} , respectively. Using Lemma 6.3 below, it is simple to see that $\tau_t(g, h)$ is defined for all $(g, h) \in G * H$ and $t \in \mathbb{R}$, whenever $\rho_t(g)$ is defined for all g and t . Since $p_{1*}\vec{X}_B = \vec{X}$, it also follows that

$$p_1 \circ \tau_t(g, h) = \rho_t(g).$$

Suppose that σ_t is any path in G_m starting from 1_m and which is a product of paths generated by right invariant vector fields \vec{X}_i , for compactly supported sections $X_i \in \Gamma(AG)$. Let $\tilde{\sigma}_t$ be the corresponding path in $G * H$, the product of the flows generated by the right invariant vector fields \vec{X}_{iB} in the same order. Clearly, $\tilde{\sigma}_t$ lies in \mathcal{L}_m , and is a lift of σ_t . Since any path in G_m can be approximated by such paths σ_t , this proves the path lifting property for p_G .

Since G_m is simply connected by assumption, it follows that $p_G: \mathcal{L}_m \rightarrow G_m$ is a diffeomorphism. Therefore it defines a smooth map $\Phi_m: G_m \rightarrow H_{f(m)}$, where $H_{f(m)} = \alpha_H^{-1}(f(m)) \subseteq H$. Since $\mathcal{L}_m \subseteq G * H$, Φ_m commutes with β_G and β_H . In fact, Φ_m is characterized by the following properties:

- (i). $\Phi_m(1_m) = 1_{f(m)}$;
- (ii). $\beta_H \circ \Phi_m = \beta_G$;
- (iii). $T_g(\Phi_m)(Y) = T(R_{\Phi_m(g)})[\varphi(T(R_{g^{-1}}(Y)))]$, for all $g \in G_m$ and $Y \in T_g G_m$.

By varying m in M , we obtain a family of smooth maps $\Phi_m: G_m \rightarrow H_{f(m)}$, and hence a global map Φ from G to H .

Fix any $y \in G$ and suppose that $m = \beta_G(y)$, $n = \alpha_G(y)$. Consider the map $\Psi: G_m \rightarrow H_{f(m)}$ given by

$$\Psi(x) = \Phi_n(xy)\Phi_n^{-1}(y), \quad \forall x \in G_m.$$

It is simple to see that Ψ satisfies the properties (i)–(iii) above. Therefore, $\Psi = \Phi_m$, which is equivalent to

$$\Phi_n(xy) = \Phi_m(x)\Phi_n(y).$$

In other words, Φ is a groupoid morphism.

It remains to prove that Φ , as a global map from G to H , is smooth. First, it is easy to see that, by construction, it is smooth along the identity space M . In fact, the graph of Φ , in a neighborhood of any $m \in M$, is the flow box in $G * H$ obtained from flowing the graph of f along the distribution \mathcal{D} , which is transversal to it; thus we obtain a smooth submanifold.

Now take any point $x_0 \in G$. As in Lemma 2.3, there exist $X_1, \dots, X_n \in \Gamma_c(AG)$ such that $\exp X_1 \dots \exp X_n$ has the value x_0 at αx_0 . For notational convenience in what follows, assume that $n = 1$, so that X has $\exp X(\alpha x_0) = x_0$.

Lemma 6.2 *The map $\lambda: M \rightarrow H$ defined by $\lambda(m) = \Phi(\exp X(m))$, $m \in M$, is smooth.*

Proof. Let γ be the graph map of f regarded as $M \rightarrow G * H$; thus $\gamma(m) = (1_m, 1_{f(m)})$. It is easy to see, from the definition, that λ is the composition $p_2 \circ \tau_1 \circ \gamma$, where τ_1 is the time-1 map of the flow τ_t introduced early in the proof, and $p_2 : G * H \rightarrow H$ is the projection. Hence λ is smooth. □

Since Φ is a groupoid morphism, for x in a neighborhood of x_0 , we have

$$\Phi(x) = \Phi(\exp X(m(x)) \cdot \exp X(m(x))^{-1} \cdot x) = \Phi(\exp X(m(x)))\Phi(\exp X(m(x))^{-1} \cdot x)$$

where $m(x) = \text{Ad}_{\exp^{-1} X}(\beta_G x)$; note that $\text{Ad}_{\exp X}$ is the time-1 map for $a(X) \in \mathcal{X}(M)$.

According to Lemma 6.2, $\Phi(\exp X(m(x)))$ is a smooth map from G to H . On the other hand, $\exp X(m(x))^{-1} = \exp -X(\beta_G x)$, and so $\Phi(\exp X(m(x))^{-1} \cdot x) = (\Phi \circ L_{\exp -X})(x)$ is also smooth in a neighborhood of x_0 . Since the groupoid multiplication is smooth, it follows that Φ is smooth in a neighborhood of x_0 . This concludes the proof of Theorem 6.1. □

Finally we recall the result of Kumpera and Spencer used above.

Lemma 6.3 ([17]) *Let G be any Lie groupoid over base M , and let AG be its Lie algebroid with anchor a . For $X \in \Gamma AG$, let \vec{X} be the right invariant vector field corresponding to X , and recall that $a(X) = \beta_* \vec{X}$ is its projected vector field on M . Then \vec{X} is complete if and only if $a(X)$ is complete. In fact, $\tilde{\sigma}_t(x)$ is defined whenever $\sigma_t(\beta(x))$ is defined, where $\tilde{\sigma}_t$ and σ_t are the flows generated by \vec{X} and $a(X)$, respectively.*

Theorem 6.1 was announced by Pradines [29]; a method of proof (based on the local integrability of Lie subalgebroids) was briefly indicated by Almeida and Kumpera [3]. When both Lie algebroids are transitive and over the same base, with the map f an identity, the proof can be reduced to standard results of connection theory: see [23, III§7], which also gives further references for this case. Providing the target Lie algebroid is transitive, the case of a general base map may be reduced to that of an identity by using a pullback [10]. When the target is a Lie algebra (so that the base map is constant), the result reduces to the integration of Maurer–Cartan forms: see [36, §5]; this method of proof can be extended to handle any transitive target. For the case of a compact base (and f an identity), a proof has been given by Mokri [28]. See also [5] for the closely related construction of the monodromy groupoid of an α -connected groupoid.

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